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# The Schröder functional equation and its relation to the invariant measures of chaotic maps

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## Abstract

The aim of this paper is to show that the invariant measure for a class of one-dimensional chaotic maps,  $T(x)$ , is an extended solution of the Schröder functional equation,  $q(T(x)) = \lambda q(x)$ , induced by them. Hence, we give a unified treatment of a collection of exactly solved examples worked out in the current literature. In particular, we show that these examples belong to a class of functions introduced by Mira (see the text). Moreover, as a new example, we compute the invariant densities for a class of rational maps having the Weierstrass  $\wp$  function as an invariant one. Also, we study the relation between that equation and the well-known Frobenius–Perron and Koopman’s operators.

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## 1. Introduction

The notion of an invariant measure plays a central role in the statistical characterization of chaotic dynamical systems. In spite of the well-established theorems about the existence of such measures for abstract dynamical systems the construction of such a measure for a given dynamical system is in general a difficult task. In the case of chaotic maps it is well known that if the invariant measure has a density, this one is a solution of the Frobenius–Perron functional equation. In one-dimensional dynamics, some rigorous results about the existence of such measures are known, see for example [4, 11, 15, 17, 20–22]. There is no doubt about the physical significance of the invariant measures, with them we can compute *ensemble* averages, correlation functions, etc, in particular modeling non-equilibrium states [5]. Therefore finding reliable analytical methods to construct these measures has special interest.

Over the last three decades a series of papers, devoted to deterministic chaos, have been reporting examples of chaotic maps that are handled by means of a change of variables method [6, 14–16, 19, 20, 22, 23, 25, 27–29]. Their study is concerned with nonlinear piecewise transformations that are topologically conjugated, or semi-conjugated, to linear piecewise maps. Hence, the invariant density for the nonlinear case is obtained by transforming the constant piecewise density of the linear one. In particular, it has been remarked in some of these papers that the invariant measure of the nonlinear map is in fact the function doing such conjugation. Also, this procedure allows us to generalize some results proved for piecewise linear maps to nonlinear maps conjugated to them. For example, existence theorems, ergodic and asymptotic properties or spectral decompositions of nonlinear transformations [5, 16, 22].

Moreover, from the experimental point of view, one-dimensional maps are useful in modeling nonlinear processes in diverse fields of science. Also, from a theoretical point of view with the study of nonlinear maps, which are conjugated to piecewise maps, we gain insight into integrable chaos [13, 23].

In this paper we provide a new approach to the problem of computing invariant densities and measures for chaotic maps, based on Schröder’s functional equation. This method becomes an alternative to the traditional one given by the Frobenius–Perron equation, but also more important, that can be interpreted as an adjoint problem to this one. The main difference between these two approaches comes from the fact that for Schröder’s equation we only need to know the transformation and its derivative, without the explicit knowledge of the inverse transformation and its respective derivative, as it is demanded for the Frobenius–Perron equation. Also, the fact that Schröder’s equation is older than the Frobenius–Perron equation is very interesting. Then, we give a new look to an old equation.

This paper is divided into seven parts, section 2 is concerned with a brief exposition of the well-known conjugation property between maps. Section 3 presents several exactly solvable examples. The fourth one presents the main result: the formal relation between the Schröder and Frobenius–Perron equations; hence, we proceed to the construction of the invariant density. In section 5, we give a new example, a class of functions having the Weierstrass  $\wp(x)$  function as an invariant. Finally, we compute Lyapunov’s exponent of these maps and discuss some possible generalizations of our method.

## 2. Conjugate maps

Let us consider a transformation  $T$  of an interval  $I$  into itself, which preserves the measure  $\mu$ , i.e.  $\mu(A) = \mu(T^{-1}(A))$ . Let  $x \in I$ , the iteration

$$x_{n+1} = T(x_n), \tag{1}$$

defines a discrete dynamical system. Now, if there is a transformation  $S : I \rightarrow I$  related to  $T$  by a change of variables  $T(h(x)) = h(S(x))$ , where  $h : I \rightarrow I$  is a continuous one-to-one function, we say that the maps  $T$  and  $S$  are topologically conjugate. If in addition,  $h$  is a piecewise differentiable function then their invariant densities are related by

$$\rho_T(x) = \frac{\rho_S(h^{-1}(x))}{|h' \circ h^{-1}(x)|}. \tag{2}$$

The simplest examples of linear piecewise transformations of an interval are the Rényi transformation  $R_r(x) := \{rx\}$  and the piecewise continuous transformations [15, 16, 22]

$$N_r(x) := (-1)^{[rx]} \{rx\}, \tag{3}$$

where  $[rx]$  is the integer part of  $rx$  and  $\{rx\}$  means  $rx \bmod 1 = rx - n$ , where  $n$  is the largest integer such that  $rx - n \geq 0$  for any  $r \in \mathbb{N}$ .

In the following we are concerned with the case when  $S$  is  $R_r$  or  $N_r$ , hence  $\rho_S(x) = 1_{[0,1]}(x)$ , for any  $r \in \mathbb{N}$ . Therefore,  $\rho_T(x) = 1/|h' \circ h^{-1}(x)|$ . Also, we denote by  $T_r$  any map conjugated to  $N_r$  or  $R_r$ . It is implied by the conjugation property that each map  $T_r$  has  $r$  monotonic pieces that map the interval  $I$  onto itself. The inverse function  $T^{-1}$  has  $r$  monotonic branches on  $I$ , each one denoted by  $T_i^{-1}$ .

### 3. Exactly solvable maps

To show the essence of our method we work out in detail the logistic transformation. It is well known that it is conjugated to  $N_2$  and that its invariant density satisfies the Frobenius–Perron equation.

#### 3.1. Logistic map

Let us consider the transformation  $F(x) = 4x(1 - x)$ , where  $x \in [0, 1]$ . The function  $h(\theta) = \sin^2(\frac{\pi}{2}\theta)$ ,  $\theta \in [0, 1]$ , satisfies

$$F[h(\theta)] = h[N_2(\theta)], \tag{4}$$

hence, by (2)

$$\rho_F(x) = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{5}$$

Finally, the invariant measure for  $F$  is

$$\mu(x) = \int_0^x \frac{1}{\pi\sqrt{y(1-y)}} dy, \quad x \in I, \tag{6}$$

$$= \frac{2}{\pi} \arcsin(\sqrt{x}), \tag{7}$$

in other words,  $\mu(x) = h^{-1}(x)$ . It should be noted that (4) can be written as  $F(x) = h[N_2(h^{-1}(x))]$  then, by periodicity of the sin function the last expression is equal to  $F(x) = h(2h^{-1}(x))$ , i.e.  $F(x) = \sin^2(2 \sin^{-1} \sqrt{x})$ .

#### 3.2. A collection of exactly solvable maps

Exactly solvable iterations have been known since the 19th century; the interest was to understand Newton’s method [6, 9, 24]. However, in the last 30 years a boom in the search of analytic results for chaotic systems shifted attention to the topic.

Very important examples are provided by the Chebyshev polynomials. It is well known that they are ergodic and mixing maps [22]. From its definition:  $T_r(x) = \cos(r \cos^{-1} x)$ , we can see that they are conjugated to  $N_r(x)$  maps. To show this, we start with the function  $h(\theta) = \cos(\pi\theta)$ , and obtain  $T_r(x) = h[N_r(h^{-1}(x))]$ . Hence, using (2),  $\rho_T(x) = 1/\pi\sqrt{1-x^2}$  such that its invariant measure is  $\mu(x) = \frac{1}{\pi} \arccos(x)$ . We can also see that  $F(x)$  is also conjugated to  $T_2(x)$  via  $N_2(x)$ .

Now we list some papers on the subject: (a) in [15] the invariant density of maps conjugated with the Jacobian elliptic sine function,  $sn(x)$ , are presented. The class of maps studied is  $T_r(x) = f(sn(rsn^{-1}(f^{-1}(x))))$ , where  $f(x)$  is an appropriate function mapping the interval  $[0, 1]$  onto itself; (b) in [19] some examples of one-dimensional iterations are solved using the addition properties of trigonometric and elliptic functions; (c) in [14] transformations of the real line associated with special statistical distributions are built. In particular, the

Cauchy,  $F$  and  $Z$  distributions are associated with the maps of the real line conjugated to  $N_2$ ; (d) in [6, 28] Newton’s algorithm to search the square root of  $\sqrt{-3}$  is found to be conjugated to  $N_2$ , or as in [27] to  $2x + 1/2 \pmod 1$ ; (e) recently, in [29] a similar method as [15] is given; (f) finally, an exception is the paper [25] where Schröder’s equation (see below) is explicitly used in order to generalize the properties of the Chebyshev polynomials to more general functions. But, in that paper the relation to the Frobenius–Perron equation is only suggested. Nevertheless, it should be noted that the invariant densities of all those examples are exactly computed exploiting the addition and periodicity properties of trigonometric and Jacobian elliptic functions jointly with the conjugation property with  $R_r$  or  $N_r$ . It is worth mentioning that the examples handled in these papers were obtained by each author mostly in an independent way to the other ones. For that reason we speak about *rediscovered* maps in the last paragraph of 5.

#### 4. Schröder’s equation

By definition, if  $\mu(x)$  has a density  $\rho(x)$ , then

$$\mu(x) = \int_0^x \rho(y) \, dy, \quad x \in I. \tag{8}$$

Now, the preservation of  $\mu$  under  $T$  implies that  $\rho(x)$  satisfies the *Frobenius–Perron* functional equation [11]:

$$\rho(x) = \sum_{i=1}^r \frac{\rho(T_i^{-1}(x))}{|T' \circ T_i^{-1}(x)|}, \tag{9}$$

or, in an integral form:

$$\rho(x) = \int_I \delta(x - T(y))\rho(y) \, dy. \tag{10}$$

Now, using this last expression, we ask the question about the left eigenvalue problem for the Frobenius–Perron equation, which is defined by

$$\lambda q(x) = \int_I q(y)\delta(y - T(x)) \, dy = q(T(x)).$$

This equation is nothing but Schröder’s functional equation [9, 18]:

$$\lambda q(x) = q(T(x)), \tag{11}$$

where  $\lambda \in \mathbb{R}$  and  $q(x)$  is a function of a real variable. Hence, the iteration (1) is linearized, such that its  $n$ th iterate is (at least locally):  $\lambda^n q(x) = q(T^n(x))$ .

The classical result in the local analysis of solutions of Schröder’s equation is Koenig’s theorem which guarantees the existence of a solution in the neighborhood of a fixed stable point of  $T$  [17, 18]. However, our present interest in the global study of that equation is more difficult, see [4, 17, 18] for a discussion.

Now we consider a differentiable map  $T : I \rightarrow I$  conjugated to  $N_r(x)$ ; then  $T$  has  $r$  monotonic pieces on  $I$ , and  $T^{-1}$  has  $r$  branches on  $I$ . We proceed as follows: taking the derivative on both sides of Schröder’s equation (11) and after taking its absolute value, we obtain

$$|\lambda| |q'(x)| = |T'(x)| |q'(T(x))|. \tag{12}$$

Indeed, substituting the  $j$  th branch of  $T^{-1}$  into (12)

$$|\lambda| |q'(T_j^{-1}(x))| = |T'(T_j^{-1}(x))| |q'(x)|. \tag{13}$$

$x \in I$ . We observe that this expression is valid for any  $j = 1, \dots, r$ . We are now ready to introduce the result of this paper, putting together all terms of the form (13):

$$|\lambda| \sum_{j=1}^r \left| \frac{q'(T_j^{-1}(x))}{T'(T_j^{-1}(x))} \right| = r|q'(x)|, \tag{14}$$

where  $j$  runs over all the inverse branches of  $T$ . As we can see, this is the Frobenius–Perron equation for  $|q'(x)|$ ; therefore, if there exists a function  $\alpha(x) = |q'(x)|$  as a global solution of (14), defined on  $I$ , for  $|\lambda| = r$ , then the corresponding invariant measure of  $T$  is

$$\mu(x) = \int_0^x |q'(y)| dy, \quad y \in I. \tag{15}$$

It is very easy to prove, by direct substitution, that the examples handled in [6, 14–16, 19, 22, 25, 27–29], which are conjugated to  $N_r$ , satisfies equation (12). For example, we use the logistic map to show the method. First, substituting  $F_{\pm}^{-1}(x) = (1 \pm \sqrt{1-x})/2$  into the lhs of (12),

$$|\lambda| \rho(F_{\pm}^{-1}(x)) = \frac{|\lambda|}{\pi \sqrt{F_{\pm}^{-1}(x)(1 - F_{\pm}^{-1}(x))}} = \frac{2|\lambda|}{\pi \sqrt{x}}, \tag{16}$$

and into the rhs of (12),

$$|F'(F_{\pm}^{-1}(x))| \rho(x) = \frac{4\sqrt{1-x}}{\pi \sqrt{x(1-x)}} = \frac{4}{\pi \sqrt{x}}, \tag{17}$$

such that, we have an equality if and only if  $|\lambda| = 2$ .

It is worth mentioning that (12) is formally different from (11). However, if the functional equation  $\alpha(T(x)) = \frac{\lambda}{T'(x)}\alpha(x)$  has a continuous solution  $\alpha(x)$  on  $I$ , fulfilling  $\lambda = T'(\bar{x})$ , then  $q(x) = \int_0^x \alpha(x) dx$  is a  $C^1$  solution of (11). Thus, there exists a one-to-one correspondence between the  $C^1$  solutions of Schröder’s equation and the continuous solutions of (12) [18].

### 5. Rational transformations on the real line

Now, as a non-trivial example we consider a class of rational transformations having the Weierstrass  $\wp(z)$  elliptic function as an invariant one. This function is defined by an infinite series, also its duplication formula is known.

The Weierstrass  $\wp(z)$  elliptic function is defined as [1]

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n} \frac{1}{(z - 2m\omega_1 - 2n\omega_2) - (2m\omega_1 + 2n\omega_2)},$$

where  $z \in \mathbb{C}$  and  $\omega_1, \omega_2$  are two numbers the ratio of which is not real, and the summation is taken over all  $m, n \in \mathbb{Z}$  but excepting the case when  $m = n = 0$  simultaneously. This is an example of a double periodic function with fundamental periods  $2\omega_1$  and  $2\omega_2$ . As it is standard, that function is parametrized with two numbers,  $g_2$  and  $g_3$ , which are called the elliptic invariants (and are functions of  $\omega_1, \omega_2$ ). We describe Schröder’s method to obtain a rational function invariant under  $\wp(x)$ : there is a duplication formula [1]:

$$\wp(2z) = -2\wp(z) + \left[ \frac{\wp''(z)}{2\wp'(z)} \right]^2,$$

such that, using the identities:  $\wp''(z) = 6\wp^2(z) - \frac{g_2}{2}$ , and  $(\wp')^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3$ , we can build a two-parametric rational function

$$T_{g_2, g_3}(z) = \frac{z^4 + \frac{g_2}{2}z^2 + 2g_3z + \left(\frac{g_2}{4}\right)^2}{4z^3 - g_2z - g_3}, \tag{18}$$

having  $\wp(z)$  as an invariant function, i.e.  $\wp(2z) = T_{g_2, g_3}(\wp(z))$ . Now, let us consider the case  $g_2, g_3 \in \mathbb{R}$ ; then  $\wp$  takes real values on the real line. The inverse function for  $\wp$  is also known [3]:

$$\wp^{-1}(u) = \int_u^\infty \frac{ds}{\sqrt{4s^3 - g_2s - g_3}}.$$

Now, if the discriminant  $\Delta = \frac{1}{16}(g_2^3 - g_3^3)$  is positive, then  $4s^3 - g_2s - g_3 = 0$  has three real and distinct roots and  $\wp$  has the period  $\omega_1 \in \mathbb{R}$ . Also, the above relations are unaltered by the choice  $z = x$ . Now, to use the Frobenius–Perron equation we must compute the inverse function of the  $T$  given by (18), and its derivative. Instead of this, we proceed to compute the solution of the corresponding Schröder’s equation associated with such  $T$ . We can see that in analogy with 3.1 that  $\wp(2\wp^{-1}(x)) = T_{g_2, g_3}(x)$ ; therefore the invariant density is given by

$$\rho(x) = \left| \frac{d}{dx} \wp^{-1}(x) \right| = \frac{1}{\sqrt{4x^3 - g_2x - g_3}}, \tag{19}$$

i.e., the eigenfunction of (11) induced by  $T_{g_2, g_3}$  and having eigenvalue  $|\lambda| = 2$ .

For example, if  $g_2 = 4$  and  $g_3 = 0$ , we have the rational transformation  $T(x) = (x^2 + 1)^2 / 4x(x^2 - 1)$ , where  $x \neq 0, \pm 1$ , then  $T(\wp(x)) = \wp(2x)$ . This example was given by Lattès in 1918 [9]. Another important example is given by

$$T(x) = \frac{4x(1-x)(1-\kappa^2x)}{(1-\kappa^2x^2)^2},$$

which satisfies the semiconjugacy relation  $T(sn^2(x)) = sn^2(2x)$ , where  $sn(x)$  is the Jacobi elliptic sine function. As it is remarked by Milnor [24], this example given also by Lattès, was in fact studied by Schröder in 1871 [26]. However, we found that it was also rediscovered in [15, 19, 29]. A different example, studied by Boole in 1872, is the linear fractional transformation

$$T(x) = \frac{ax + b}{cx + d}, \quad ad - bc = 1, \quad c \neq 0,$$

whose Schröder’s equation is solved using an iterative method, called by him as Laplace’s method, which it is exposed in [7]. For this example, our method gives  $\rho(x) = 1/(1 + x^2)$  as its invariant density, provided  $|a + d| > 2$ . Again, this is a case of a rediscovered example [28].

### 6. Lyapunov’s exponents

Finally, the Lyapunov exponent  $\Lambda(T)$  of the maps  $T_r$  is computed using Schröder’s equation induced by it. By definition,  $\Lambda(T) := \int_I \ln|T_r'(x)|\rho(x) dx$ , then taking the logarithm of (12), and using the invariance of  $\mu$  under  $T_r$ , we have from the previous equation the expected result  $\Lambda(T) = \ln r$ . In other words,  $\Lambda(T)$  is equal to the logarithm of the number of monotonic pieces of  $T_r(x)$ .

### 7. Discussion and conclusions

To the best of our knowledge, we are the first to point out the significance of Schröder’s equation to compute invariant densities and measures for chaotic maps. Our approach makes the main difference with the corresponding work of the authors in [6, 14–16, 19–23, 25, 27–29]. As is mentioned in 3.2, the work by Mira [25] using Schröder’s equation, is focused on the generalization of the properties of Chebysev maps. It should be noted

the interest of authors working on the subject of functional equations on the problem of computing invariant densities, see for example [4, 17, 21]. On the other hand, the authors working in dynamical chaos are interested in methods to solve the Frobenius–Perron functional equation [5, 6, 11, 13–16, 19–23, 25, 27–29]. However, in the current literature the problem of computing invariant densities or measures appears unrelated to the problem of solving Schröder’s functional equation.

We remark that the measure  $\mu(x)$ , which is the integral of  $|q'(x)|$  in (15), could not be equal to  $q(x)$ , is in the sense that we speak about an *extended* solution of Schröder’s equation.

Also, we note that in Ergodic theory, Koopman’s operator is introduced as the formal adjoint of the Frobenius–Perron operator on densities in the  $L^\infty$  sense [11, 22, 23]. In our context, we are considering Schröder’s equation as a left eigenvalue problem associated to the Frobenius–Perron functional equation, such that our case is a more general one. It should be pointed out that this investigation is not limited to the one-dimensional case. However, higher dimensional dynamics seems to be more difficult and may be requiring a different approach, mainly due to the formal definition of Schröder’s equation in several variables [12].

In this paper we are mainly concerned with real transformations on the interval, but another interesting aspect of the subject is the study of the complex Schröder’s equation associated with chaotic transformations of the complex plane [8, 9, 12, 27].

Finally, in the one-dimensional case, the study of strange attractors at  $2^n$  banded chaos of the logistic map  $F_a(x) = ax(1 - x)$  becomes very interesting. Here  $a \in (a_\infty, 4)$ , and  $a_\infty$  is the accumulation point of the period-doubling bifurcation [16, 22, 23]. Further studies in this direction will be published elsewhere.

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